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# EXTERIOR DIFFERENTIALS IN SUPERSPACE AND POISSON BRACKETS OF DIVERSE GRASSMANN PARITIES*

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It is shown that two definitions for the exterior differential in superspace, giving the same exterior calculus, when applied to the Poisson bracket lead to the different results. Examples of the even and odd linear brackets, corresponding to semi-simple Lie groups, are given and their natural connection with BRST and anti-BRST charges is indicated.

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**1.** There exist two possibilities to define an exterior differential in superspace with coordinates $z^a$ having Grassmann parities $g(z^a) \equiv g_a$ and satisfying the permutation relations

$$z^a z^b = (-1)^{g_a g_b} z^b z^a. \tag{1}$$

The first one realized when we set the Grassmann parity of the exterior differential $d_0$ to be equal to zero $g(d_0 z^a) = g_a$ and a symmetry property of an exterior product of two differentials is

$$d_0 z^a \wedge d_0 z^b = (-1)^{g_a g_b + 1} d_0 z^b \wedge d_0 z^a. \tag{2}$$

Another possibility arises when the parity of the differential $d_1$ is chosen to be equal to unit $g(d_1 z^a) = g_a + 1$ and the symmetry property of the exterior product for two differentials is defined as

$$d_1 z^a \wedge d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b \wedge d_1 z^a. \tag{3}$$

In this case the symmetry properties of the exterior and usual products of two differentials coincide

$$d_1 z^a d_1 z^b = (-1)^{(g_a+1)(g_b+1)} d_1 z^b d_1 z^a. \tag{4}$$

The equivalence of the exterior calculi obtained with the use of the above mentioned different definitions for the exterior differential can be established as a result of the direct verification by taking into account relations (2) and (3).

**2.** However, we show that application of these two differentials leads to the different results under construction from a given Poisson bracket with a Grassmann parity $\epsilon = 0, 1$

$$\{A, B\}_\epsilon = A \overleftarrow{\partial}_{z^a} \overset{\epsilon}{\omega}{}^{ab}(z) \overrightarrow{\partial}_{z^b} B \tag{5}$$

of another Poisson bracket (7).

Indeed, the corresponding to a Hamiltonian $H_\epsilon$ ($g(H_\epsilon) = \epsilon$) Hamilton equations for the phase variables $z^a$

$$\frac{dz^a}{dt} = \{z^a, H_\epsilon\}_\epsilon = \overset{\epsilon}{\omega}{}^{ab} \partial_{z^b} H_\epsilon \tag{6}$$

and the equations for their differentials $d_\zeta z^a \equiv y^a_\zeta$ ($\zeta = 0, 1$), obtained by differentiation of this Hamilton equations, can be reproduced by the following bracket of the Grassmann parity $\epsilon+\zeta$

$$(F,G)_{\epsilon+\zeta} = F(\overleftarrow{\partial}_{z^a}\ \overset{\epsilon}{\omega}^{ab}\ \overrightarrow{\partial}_{y^b_\zeta} + (-1)^{\zeta(g_a+\epsilon)}\ \overleftarrow{\partial}_{y^a_\zeta}\ \overset{\epsilon}{\omega}^{ab}\ \overrightarrow{\partial}_{z^b} + \overleftarrow{\partial}_{y^a_\zeta}\ y^c_\zeta \partial_{z^c} \overset{\epsilon}{\omega}^{ab}\ \overrightarrow{\partial}_{y^b_\zeta})G \tag{7}$$

with the help of the Hamiltonian $y^a_\zeta \partial_{z^a} H_\epsilon$. In the case $\zeta=1$, due to relations (3), (4), the terms in the decomposition of a function $F(z^a, y^a_1)$ into degrees $p$ of the variables $y^a_1$ can be treated as $p$-forms and the bracket (7) can be considered as a Poisson bracket on $p$-forms such that being taken between a $p$-form and a $q$-form results in a $(p+q-1)$-form. The bracket (7) is a generalization of the bracket introduced in [1] on the superspace case and on the case of the brackets (5) with arbitrary Grassmann parities.

**3.** Now we apply this procedure to the linear even and odd brackets connected with a semi-simple Lie group having structure constants $c_{\alpha\beta}{}^\gamma$. By taking as an initial bracket (5) the linear even bracket given in terms of the commuting variables $x_\alpha$ (here $z^a = x_\alpha$)

$$\{A,B\}_0 = A\ \overleftarrow{\partial}_{x_\alpha}\ c_{\alpha\beta}{}^\gamma x_\gamma\ \overrightarrow{\partial}_{x_\beta}\ B \tag{8}$$

and using the odd exterior differential $d_1$, we obtain in conformity with the transition from (5) to (7) the following odd linear bracket

$$(F,G)_1 = F(\overleftarrow{\partial}_{x_\alpha}\ c_{\alpha\beta}{}^\gamma x_\gamma\ \overrightarrow{\partial}_{\theta_\beta} + \overleftarrow{\partial}_{\theta_\alpha}\ c_{\alpha\beta}{}^\gamma x_\gamma\ \overrightarrow{\partial}_{x_\beta} + \overleftarrow{\partial}_{\theta_\alpha}\ c_{\alpha\beta}{}^\gamma \theta_\gamma\ \overrightarrow{\partial}_{\theta_\beta})G, \tag{9}$$

where $\theta_\alpha = d_1 x_\alpha$ are Grassmann variables. This bracket has a nilpotent Batalin-Vilkovisky $\Delta$-operator [2]

$$\Delta = -\frac{1}{2}[\partial_{x_\alpha}(x_\alpha,\ldots)_1 - \partial_{\theta_\alpha}(\theta_\alpha,\ldots)_1] = (T_\alpha + \frac{1}{2}S_\alpha)\partial_{\theta_\alpha}, \quad \Delta^2 = 0, \tag{10}$$

where

$$T_\alpha = c_{\alpha\beta}{}^\gamma x_\gamma \partial_{x_\beta}, \quad S_\alpha = c_{\alpha\beta}{}^\gamma \theta_\gamma \partial_{\theta_\beta}, \quad Z_\alpha = T_\alpha + S_\alpha = (\theta_\alpha,\ldots)_1 \tag{11}$$

are generators of the Lie group in the co-adjoint representation which obey the commutation relations

$$[T_\alpha, T_\beta] = c_{\alpha\beta}{}^\gamma T_\gamma, \quad [S_\alpha, S_\beta] = c_{\alpha\beta}{}^\gamma S_\gamma, \quad [T_\alpha, S_\beta] = 0. \tag{12}$$

By taking as an initial bracket (5) the linear odd bracket given in terms of Grassmann variables $\theta_\alpha$ [3] (in this case $z^a = \theta_\alpha$)

$$\{A,B\}_1 = A\ \overleftarrow{\partial}_{\theta_\alpha}\ c_{\alpha\beta}{}^\gamma \theta_\gamma\ \overrightarrow{\partial}_{\theta_\beta})B, \tag{13}$$

with the help of the differential $d_1$ we come to the even linear bracket of the form

$$(F,G)_0 = F(\overleftarrow{\partial}_{\theta_\alpha}\, c_{\alpha\beta}{}^{\gamma}\theta_\gamma\, \overrightarrow{\partial}_{x_\beta} + \overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^{\gamma}\theta_\gamma\, \overrightarrow{\partial}_{\theta_\beta} + \overleftarrow{\partial}_{x_\alpha}\, c_{\alpha\beta}{}^{\gamma}x_\gamma\, \overrightarrow{\partial}_{x_\beta})G, \qquad (14)$$

where $x_\alpha = d_1\theta_\alpha$ are commuting variables. This bracket instead of the nilpotent second order differential $\Delta$-operator has a nilpotent differential operator of the first order

$$Q = \frac{1}{2}[\theta^\alpha(x_\alpha,\ldots)_0 - x^\alpha(\theta_\alpha,\ldots)_0] = \theta^\alpha(T_\alpha + \frac{1}{2}S_\alpha), \quad Q^2 = 0. \qquad (15)$$

If we consider $\theta^\alpha$ and $\partial_{\theta^\alpha}$ as representations for the ghosts and antighosts respectively, then $Q$ and $\Delta$ can be treated as the BRST and anti-BRST charges correspondingly (see, e.g., [4]) and satisfy the anticommutation relation

$$\{Q,\Delta\} = \frac{1}{2}(T^\alpha T_\alpha + Z^\alpha Z_\alpha), \qquad (16)$$

two terms in the right-hand side of which, because of the commutation relations

$$[T^\alpha T_\alpha, Q] = 0, [T^\alpha T_\alpha, \Delta] = 0, [Z_\alpha, Q] = 0, [Z_\alpha, \Delta] = 0, [T^\alpha T_\alpha, Z_\beta] = 0, \qquad (17)$$

are central elements of the Lie superalgebra formed by the quantities $Q$, $\Delta$, $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$. $Z^\alpha Z_\alpha$ contains the term

$$D = \theta^\alpha \partial_{\theta^\alpha}, \qquad (18)$$

that can be considered as a ghost number operator which has the following relations with $Q$ and $\Delta$:

$$[D,Q] = Q, \quad [D,\Delta] = -\Delta \qquad (19)$$

and commutes with the central elements $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$

$$[D,T_\alpha] = 0, \quad [D,Z_\alpha] = 0. \qquad (20)$$

Note that the Lie superalgebra for the quantities $Q$, $\Delta$, $D$, $T^\alpha T_\alpha$ and $Z^\alpha Z_\alpha$ determined by the relations (10), (15)–(20) can be used for the calculation of the BRST operator cohomologies [5].

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## References

1. M.V. Karasev and V.P. Maslov, *Non-linear Poisson brackets. Geometry and quantization*, (Moscow, Nauka, 1991).
2. I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* B **102**, 27 (1981).
3. V.A. Soroka, *Phys. Lett.* B **451**, 349 (1999).
4. J.W. van Holten, *Nucl. Phys.* B **339**, 158 (1990).
5. S.S. Horuzhy and A.V. Voronin, *Teor. Mat. Fiz.* **93**, 342 (1992).